Stevens.

Alexander Ziwet

ANALYTICAL DYNAMICS

BEING A

SYNOPSIS OF LEADING TOPICS

——IN THE——

Analytical Theory of Dynamics with Numerous Examples and Selections from Newton's Principia and Other Sources.

—BY—

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Professor of Mathematics, Rose Polytechnic Institute,
TERRE HAUTE, INDIANA.

1906

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FIG. 1.

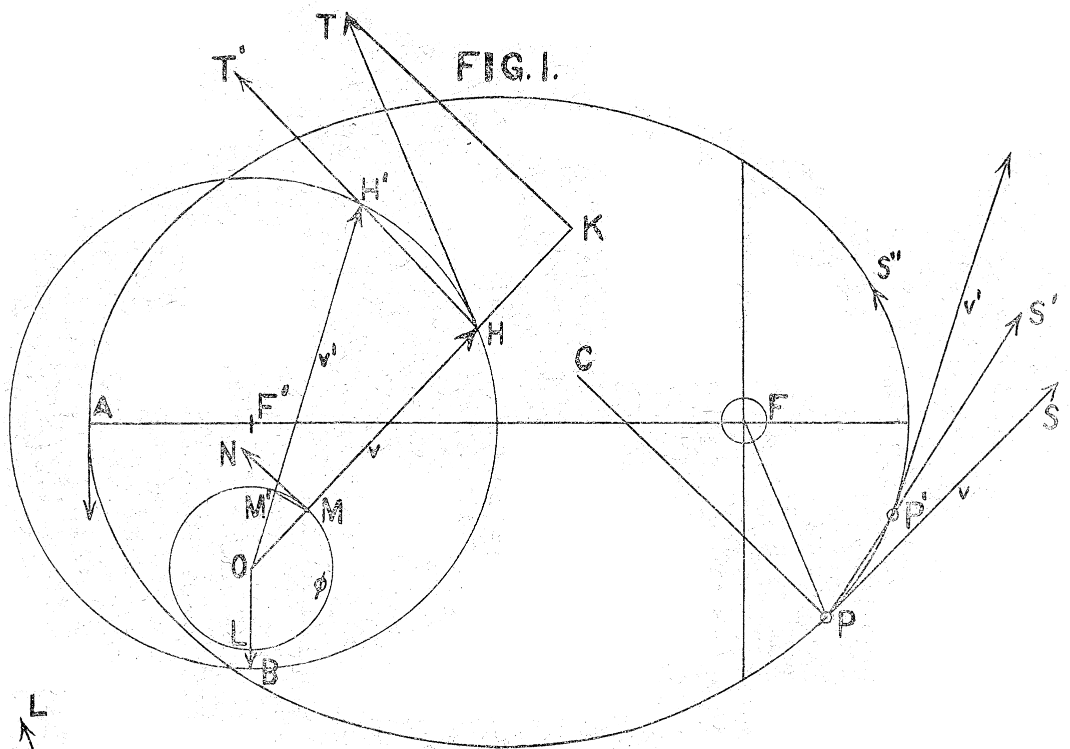


FIG. 2.

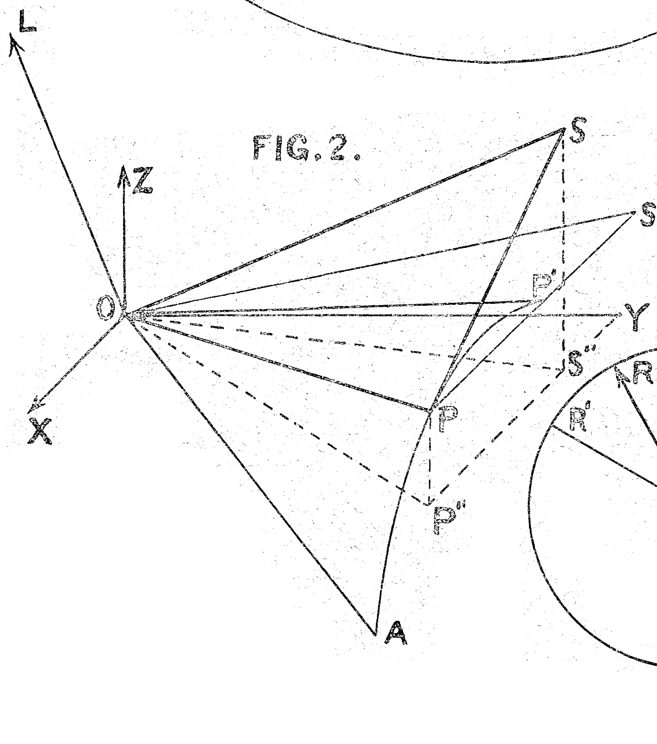
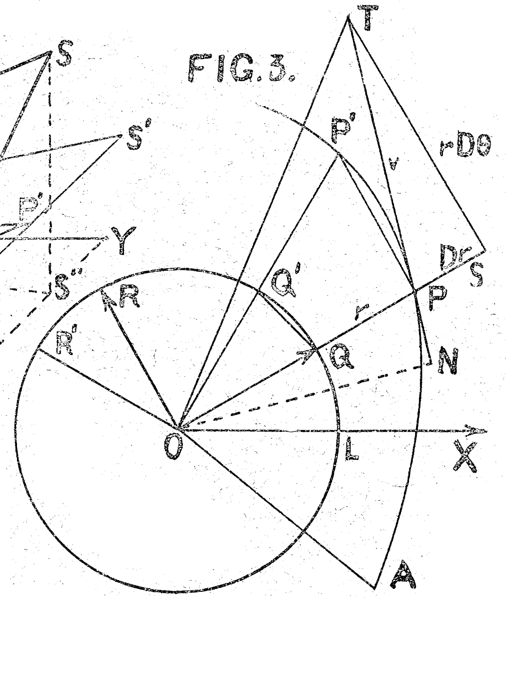


FIG. 3.



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Notes and Examples on Dynamics

KINEMATICS

We consider a point, moving on a given curve called its *path*. At time t (*fig.* 1), its arc distance from a fixed point A will be arc $AP = s$, at time t' it will be arc $AP' = s'$, and so on, changing continuously as the time changes. In other words, s is a continuous function of t, and s' is the same function of t'.

SPEED AND VELOCITY

The *average speed* from time t to time t', is the average distance passed over per unit time, or the fraction, $(s'-s):(t'-t)$.

The *speed* at time t is the limit of the above average speed. (Understanding that t' approaches t as a limit). It is denoted by v, and by definition,

(1) $v = ds/dt = Ds$ (D stands for d/dt, here and hereafter).

The speed is positive when the arc distance is increasing, and negative when the arc distance is decreasing.

The *average velocity* from time t to time t' is the average *displacement* in space per unit time, in *direction* and *magnitude*. If the point be at P (fig. 1) at time t, and at P' at time t', then the average velocity is represented by the line PS' which is the chord PP' extended in the ratio $1:(t'-t)$. On the contrary, the average speed is represented by the arc PS'' which is the arc PP' extended on the path in the ratio $1:(t'-t)$.

The *velocity* at time t is the limit of the above average velocity. Since the limit of a vanishing chord which is kept indefinitely produced, is the tangent line, the velocity at P is represented by a line PS, on the tangent to the path at P.

The velocity PS determines *the direction of motion at P*, i. e. the ultimate direction from P to a succeeding position P'. Also, its numerical measure is the speed v. For the vanishing arc and chord PP' have a limiting ratio of unity, so that the speed and velocity, which are limits of proportionals to them, will be in that ratio, as to length, or be equal lengths (the former measured properly on the path as a positive or negative distance, the latter on the tangent, as a displacement in space, but both in the same direction

from P). Remembering that the velocity is *tangential,* it is therefore completely determined by the speed v (the path and position P being known).

The term velocity as commonly used means either speed or velocity, according to the context. For motion along a given path the "compound" of two speeds is their ordinary sum, but velocities derived from displacements combine by the parallelogram law of displacements, and not by numerical addition, since they *are* displacements per unit time.

ACCELERATION—THE HODOGRAPH

The *average acceleration* from time t to time t' is the average increase of *velocity* per unit time.

The *acceleration* at time t is the limit of the above average acceleration.

From a fixed point O (fig. 1) draw lines OB, OH, OH', etc., representing, in direction and magnitude, the velocities of the point at A, P, P', etc., (and so parallel to the tangents to the path at those points, and of lengths representing the speeds at those points, V, v, v', etc.) The curve $B \ldots H \ldots H' \ldots$, is called the *hodograph* of the given motion.

Now HH' is the resultant of the velocities OH' and $-OH$ i. e., the increase of velocity from time t to time t'; and if it be produced to T' in the ratio $1 : (t'-t)$, HT' will represent the above average acceleration; and it will also represent *the average velocity of the corresponding point in the hodograph,* between the same times. Thus its limit HT, on the tangent to the hodograph at H, represents simultaneously the acceleration at P, and the velocity at H. In words,

The acceleration of a moving point is the velocity of the corresponding point in its hodograph.

TANGENTIAL AND NORMAL ACCELERATIONS

In fig. 1, draw the perpendicular, TK, to OH produced. Then HK and KT are called the *tangential* and *normal* accelerations, at P, because they are the components of the whole acceleration HT, *parallel* and *perpendicular* to the tangent at P. The normal acceleration KT, drawn from P, must lie towards the center of curvature, C, since it is in the direction in which the velocity PS begins to turn as the moving point leaves P, or towards the concave side of the path.

We proceed to find the measures of the tangential and normal accelerations.

Lay off on the velocities $OB, \ldots OH, \ldots OH' \ldots$, their corre-

sponding positive units $OL, \ldots OM, \ldots OM' \ldots$, i. e. such that $OB = V.OL$, $OH = v.OM$, $OH' = v'.OM'$. This gives a curve LMM' of unit radius, whose arc measures from L as origin are, arc $LM = \phi$, arc $LM' = \phi'$.

These are the radian measures of the angles through which the tangent to the path has turned, in the motion of its point of contact from A to P and to P'.

The curve LMM' (on a sphere of unit radius and center O), may be called *the angular directrix* of the path, and ϕ, the *angular direction* of the path at P.

The *angular velocity* of the direction of motion PS, is the velocity of the corresponding point M on its angular directrix; and is represented at time t by a tangent MN at M, whose numerical measure is $d\phi/dt$, the *angular speed.* (Remember that the treatment of velocity and speed on this curve is the same as on any other, ϕ being the arc distance of the moving point at time t.)

Draw on fig. 1, HH'' parallel to MM' to intersect OH' in H''. Then, HH' is the resultant of HH'' and $H''H'$, *i.e.* of $v.MM'$ and $(v'-v)OM'$. Hence, dividing by $t'-t$, to obtain HT' and its components, and thence, their limits, HT, and vMN, $Dv.OM$, the latter must be KT, HK, respectively, or,

$HK = Dv.OM, \quad KT = vMN.$

Remembering that the measure of OM is unity, and of MN, $d\phi/dt$, we have:

(2) *Tangential acceleration* $= f = Dv = dv/dt.$

(3) Normal acceleration $= v d\phi/dt = v^2 d\phi/ds$ $= v^2/R$, where $R = ds/d\phi$, the radius of curvature (PC) of the path. (Calc. Art. 83.)

Note. The above is *differentiation of the velocity* OH; D being a special differential symbol whose multiplying factor is $n = 1/(t'-t)$, so that $Dt = lim.n(t'-t) = 1$.

Thus let, $OM = u$, and denote *resultant* addition and subtraction by $+$ and $-$.

Then,

$$D(vu) = lim.n(v'u'-vu) = lim.n(v'u'-vu'+vu'-vu)$$
$$= lim.[n(v'-v).u'+nv(u'-u)] = Dv.u+vDu.$$

and $Du = MN$, the angular velocity.

MOTIONS IRRESPECTIVE OF PATH

The character of a motion, irrespective of path, is determined by the function which s is of t. This function may be given explicitly, or implicitly by sufficient equations between s and t and other variables for the explicit solution of s in terms of t. From

such value of s we find at once by differentation, $v=Ds$, and $f=Dv=D^2s$. These may be written

(1) $ds=vdt$, (2) $dv=fdt$, from which, (4) $vdv=fds$.

Conversely, (1) and (2) constitute two independent equations between the four variables t, s, v, f. Since one of these variables is independent, there must be three independent equations between them. The third equation is sought in the form $s=Ft$. ($F=$ "function of"); but it may be given by the problem as an equation between all four variables or less. In such a case, integrations are to be performed, which introduce arbitrary constants. These constants are to be determined by some assumed *starting conditions*, i. e., an assigned value of s, and v, at a given time t (usually time zero.) We outline methods in certain cases.

(a) When f is given in terms of t.

Find v in terms of t by integrating $dv=fdt$; thence find s in terms of t by integrating $ds=vdt$.

(b) When f is given in terms of v.

Find t and s, each in terms of v, by integrating $dt=dv/f$, $ds=vdv/f$.

(c) When f is given in terms of s.

Find v^2, and thence v, in terms of s, by integrating $2vdv=2fds$; thence find t in terms of s by integrating $dt=ds/v$.

Since $v=Ds$, $f=D^2s$, the most general analytical equation between the four variables is a differential equation of second order, for s in terms of t. This is called *the differential equation* of the motion. The equation for s in terms of t is *the integral equation* of motion. We consider some types of motion, with their integral and differential equations, leaving the verification as exercises. Notice that the integral equation always contains two constants not in the differential equation. These are consequences of two integrations.

TYPES OF MOTION

(a) *Uniform Motion.* Equal distances are passed over in equal times. $s=at+b$. $D^2s=0$.

(b) *Uniformly Increasing Motion.* Equal speeds are gained in equal times. $s=\frac{1}{2}at^2+bt+c$, $D^2s=a$.

(c) *Geometric Motion.* From a given origin, called *the center*, arc distances at successive *equally different times* are in geometrical progression. $s=a.\exp(nt)+b$, $D^2s=nDs$.

Note exp (x)=that power of the naperian base $e=2.71828+$ whose exponent is x. $d\exp(x)=exp(x).dx$, and the same for D.

(d) *Harmonic Motion.* The motion of a point on a diameter of a circle which is the orthogonal projection of a point moving *uniformly on the circumference.*

$s-c=a\cos(nt+b)$, *or* $a\sin(nt+b')$, or $A\cos(nt)+B\sin(nt)$ [equivalent forms.]

$D^2s+n^2(s-c)=0$.

The center is $s=c$. The radius or *amplitude* is $a=\sqrt{(A^2+B^2)}$ The *angular speed* is n, the period, $2\pi/n$. The *phase angle* is $b=b'-\frac{1}{2}\pi=\tan^{-1}(B/A)$.

(*e*) *Hyperbolic Motion.* The "harmonic" functions above replaced by "hyperbolic" functions. This includes geometric motion, and is in general the compound (sum) of two geometric motions of reciprocal ratios, as in the second form.

$s-c=A\cosh(nt)+B\sinh(nt)$, or $a\exp(nt)+b\exp(-nt)$ &c.

$D^2s=n^2(s-c)$.

(*f*) *Harmonic Motion of Geometrically Decreasing Amplitude.*

$s-c=a\exp(-\frac{1}{2}kt).\cos(nt+b)$.

$D^2s+kDs+(n^2+\frac{1}{4}k^2)(s-c)=0$.

Note. To prove that the integral equation includes every solution of the differential equation, let $s=x.\exp(-\frac{1}{2}kt)+c$, whence x is any solution of $D^2x+n^2x=0$.

FIELDS AND MEDIUMS

In nature, a moving point is associated with matter, and because of the inertia property of matter, cannot be conceived as in itself changing its state of rest or of uniform motion in a straight line. In this view of the motion of a point, it will be called *a particle.* The motion of a particle is simply the geometrical motion of a point, and its velocity and acceleration are geometrical quantities only, but we regard the acceleration, or change of velocity per unit time, as being produced by some cause outside the particle.

Thus, a particle near the surface of the earth is given an acceleration g, downward, wherever it is placed, whose effect is only anulled by some other cause which gives the particle the opposite acceleration, so that the resultant acceleration of the particle is zero.

A space in which a particle is given an acceleration which varies only as its position varies, will be called *a field.* In a field the acceleration, in direction and magnitude, is the same at the same point at all times and becomes the acceleration of the particle when it arrives at that point.

In a *constant field,* the acceleration is the same in magnitude and direction at all points of the field. It will be suggestive to call its direction *downward,* and to illustrate the magnitude, g, by the numerical value 32, or 981, in the foot second, or centimeter second system.

In a *central field,* the acceleration is at all points either towards or from a *fixed* point called *the center* of the field. Practically the earth's field is such in which the acceleration is towards the center of the earth and inversely as the square of the distance from the center. When the acceleration depends upon the distance from the center only, the *absolute acceleration* is the acceleration at *units distance* from the center.

We may consider *a uniform medium,* which retards speed, but produces no change in the direction of motion. Its acceleration is therefore *wholly tangential,* and of *opposite sign to the speed;* it should be in magnitude, some function of the speed alone, which vanishes with the speed, and increases as the magnitude of the speed increases. *The absolute retardation* of a medium is its retardation for *unit* speed, and is considered a function of the density of the medium which increases with increasing density. It is often assumed *proportional* to the density of the medium.

CONSTRAINED PATHS. FRICTION.

A particle in any field may describe any path, regarded as a tube in which the particle is constrained to move. The tube serves, in effect, as a producer of acceleration *in addition to the field,* since its confining walls change the direction of the velocity, and that is acceleration when measured per unit time. We generally consider that this acceleration is *wholly normal,* i.e., not affecting the speed. Also, the side of the tube which produces the acceleration by "pressing" upon the particle is that through which the particle would pass, if the field alone acted. All other sides but the one exerting pressure may be removed without affecting the motion on the path.

The normal acceleration produced by the pressure of a tube combines with the normal acceleration of the field to produce the normal acceleration of the particle (v^2/R); *i. e.,* it is the resultant of the normal acceleration v^2/R and the *reversed* normal acceleration of the field.

The tangential acceleration which we conceive the friction of a *rough* path to produce, is always opposed to the motion. It is treated as *constant,* or *in a constant ratio to the normal acceleration of the path,* called *the coefficient of friction.* The whole retardation of friction is called into play only on a particle in motion, or on the point of moving.

EXAMPLES I.

1. If the tangential acceleration is always zero, the motion is uniform; and conversely. If the normal acceleration is always

zero, the path is a straight line; and conversely. If the whole acceleration is always zero, the point moves uniformly in a straight line; and conversely.

2. If a particle move freely in a field of constant acceleration, show that the hodograph motion is uniform motion in a straight line in the direction of the field.

3. A shot is fired at a velocity of 320 feet per second at an inclination whose slope is 4/3. Draw the hodograph of the subsequent motion of the shot to a scale in which an inch represents 64 units of speed, and determine by it the velocity at the end of successive seconds, for the first sixteen seconds. Show that the greatest height is reached at the end of eight seconds.

4. Figure 1 is the elliptic path of a planet, P about the sun at the focus F; and its hodograph is a circle. Sketch on the figure the velocity of the planet at various points, including the ends of the axes and focal ordinates. Also sketch the direction of the acceleration of the planet at each point.

5. Show that in the speed-curve of any motion the space described between the times of two abscissas is represented by the area of the curve between the corresponding ordinates. Also, that the *tangential* acceleration at the time of any abscissa is represented by the slope of the curve at the point corresponding to that abscissa.

Note. The *speed-curve* of a motion is the locus of a point in the coordinate plane, whose abscissa represents the time, and ordinate, the speed. It furnishes a representation of all four variables, t the abscissa, v the ordinate, s the ordinate area, f the slope.

6. Show that the speed-curve of constant tangential acceleration is a straight line whose slope is the acceleration, and y-intercept, the speed at time zero.

7. Draw the speed-curve of the semi-cubical motion $s=8t^{3/2}$.

8. A point begins with speed V, and moves with constant tangential acceleration g. If s be the distance passed over, and v, the speed acquired, at the end of t units of time, show by the calculus, and by the speed-curve, that

$$v=gt+V, s=\tfrac{1}{2}gt^2+Vt, \quad v^2-V^2=2gs.$$

9. Show that with constant tangential acceleration, the average speed is the arithmetical mean of the initial and final speeds.

10. Find the motion whose average speed is the geometric mean of the initial and final speeds.

Note. If a be the initial speed, $s=\sqrt{(av)}\ t$. Squaring and putting ds/dt for v, this becomes $ads/s^2=dt/t^2$.

Ans. $s=at/(1+ct)$

11. Find the space passed over in any given time by a particle moving with a speed which varies as the square of the time.

12. Find the space described from rest at any time by a point whose acceleration varies as the m' th the power of the time.

13. Find the speed acquired from rest, when a particle is accelerated towards a center proportional to the distance. Find the time of describing a given space, and the time to reach the center.

14. If the tangential acceleration of a motion vary inversely as the speed, the square of the speed varies uniformly with the time, and the cube of the speed, uniformly with the arc distance.

15. Determine the motion when the space described in any time, varies as the speed acquired.

16. Show that the compound of two harmonic motions of the same period is a harmonic motion of that period.

17. Show the following construction for the distance of a point in harmonic motion from its center at any time.

Draw an initial line OX, an angle AOX, minus the phase angle b, on which $OA=a$, the amplitude, and describe the circle on OA as diameter. Then a line coinciding with OX at time zero and revolving about O, uniformly in the period of the motion, cuts this circle at any time in a point P such that OP is the required distance from the center at that time. OA is called *the vector* of the harmonic motion.

18. OA and OB are the vectors of two harmonic motions of the same period. Show that the vector of their compound is OC, where $OACB$ is a parallelogram.

PROBLEMS OF A CONSTANT FIELD

19. If a particle is constrained to move on a line inclined at an angle i to the horizontal, show that its tangential acceleration is $g\sin i$ down the line, and that the constraint of the line must produce a normal acceleration $g\cos i$. The line must lie immediately beneath, not above the particle.

20. Show that the time of descent of a particle down any chord of a vertical circle, beginning at the highest point, is constant, and the same as the *time* of descent down any chord terminating in the lowest point of the circle.

21. The right line of quickest descent in a vertical plane from a given point to the circumference of a lower circle in the plane, passes through the lowest point of the circle. The right line of quickest descent from the circumference to a lower point in its plane, passes through its highest point.

22. Find the right line of quickest descent between two circles in the same vertical plane.

23. If the projection of a right line on a vertical line be constant, show that the time of descent on that line varies as the length of the line, and that the speed acquired is constant.

24. A plane is inclined at an angle i to the horizon, and contains a line that makes an angle a with the intersection of the plane with the horizon. Prove that the acceleration down this line is $g\sin i\ \sin a$.

25. A particle moves in a circle in a vertical plane. Prove that its tangential acceleration is $g.\sin\phi$, where ϕ is the angle between the vertical and the radius to the point. Thence express f in terms of s, measured from the lowest point of the circle, the radius being l. Ans. $f=-g.\sin(s/l)$

26. If in the preceding example, the particle start from rest at $s=a$, find v in terms of s, and express t as a definite integral between the limits a and s.

27. The Simple Pendulum. If in the preceding example, the arc described be small, show that the motion is practically the harmonic motion $s=a.\cos nt$, where $n^2=g/l$. The period is consequently $2\pi\sqrt{(l/g.)}$

Note. For small arcs $\sin(s/l)=s/l$, practically.

28. Calculate the length of a second's pendulum (period 2 seconds.)

29. A pendulum beating seconds is 39.14 inches in length. The bob is screwed up one turn, the screw being 32 threads to the inch. Show that it will gain 34.7 seconds in a day in consequence.

30. A particle moves on a given curve (freely or by constraint.) Its arc distance from a fixed point being s, and its height above a given horizontal plane being y, at time t, show that the tangential acceleration is $f=-gdy/ds$. *Hence* from (4), if its height is h, and speed V, at the start, then at time t, $v^2=V^2+2g(h-y)$. In particular if the particle fall vertically from *rest* $V^2=2g(h-y)$, where $h-y$ is the distance fallen.

31. Show from the result of the preceding example, that a particle in motion on any path has at any time the same speed that it would acquire by falling vertically from a fixed horizontal plane to its given position.

Note. Let k be the height of a fixed horizontal plane above the starting position such that $2gk=V^2$. Then $v^2=2g(k+h-y)$ and $k+h-y$ is the height of that plane above the particle. That plane is *the level of zero speed* for the motion, and the particle cannot rise above it, since v^2 must be a positive quantity. The path may not rise to that level, in which case the speed cannot become zero.

32. Consider a cycloid generated by a point on a circle in a

vertical plane, which rolls on its higher horizontal tangent. With horizontal and vertical axes of x and y, through the lowest point, show that its equation is,

$x=a(2\phi+\sin 2\phi)$, $y=a(1-\cos 2\phi)$

where a is the radius of the generating circle, and 2ϕ is the angle which the diameter through the tracing point (x,y) has turned from its initial position at the vertex.

Also, s being its arc from the vertex, show that $ds=4a\cos\phi d\phi$, $s=4a\sin\phi$, or $s^2=8ay$.

Also, $dy/dx=\tan\phi$, whence ϕ is the angle through which the tangent to the cycloid has turned from the vertex, and $ds/d\phi$ is the radius of curvature at the point (x,y).

33. If a particle is constrained to move in the above cycloid, show that its motion is harmonic, with the vertex as center, and angular velocity, $\frac{1}{2}\sqrt{(g/a)}$. (Use $s^2=8ay$, and ex. 30). Deduce thence, the result of ex. 18, by assuming the small circular arc practically a small arc of a cycloid of the same curvature as the circle at its lowest point.

34. Find the speed with which the above circle must roll on its upper tangent in order that a point on its circumference may execute the motion of a particle sliding on the cycloid which it generates.

35. In the preceding motion of a particle, show that the constraint of the cycloidal path produces an acceleration of $2g\cos\phi$ towards the point of contact of the rolling circle.

36. *The conical pendulum.* A particle is on the inner surface of a smooth sphere, the horizontal small circle through its position being l units below the center of the sphere, and having a radius R. Show that if it is started in this small circle with speed $v=R\sqrt{(g/l)}$, and for that speed only, it will describe the small circle as a path, and with uniform motion whose time of revolution is the period of a simple pendulum of length l.

Note. Consider the small circle to be a smooth tube. A particle started in it with any speed will move uniformly, since there is no change of level (ex 31). The resultant constraint at any point must produce an acceleration v^2/R towards the center of the circle, and an upward acceleration balancing gravity. If this resultant is towards the center of the sphere, the tube may be removed without affecting the motion, because the sphere alone will produce the needed restraint.

37. What is the least horizontal velocity with which the above particle may be started at the lowest point of the sphere so that it will describe a vertical great circle.

Ans. With speed due to the zero level 3/2 the radius above the center.

38. If in the above example, the zero level of the starting speed is n times the radius above the center, where n is less than $3/2$, show that the particle will leave the spherical surface at an arc distance from the top whose radian measure is $\cos^{-1}(2n/3)$.

39. A particle on the outside of a smooth sphere, at its highest point, is started horizontally with a speed whose zero level is n times the radius above the center. Show that if $n=3/2$ or greater, the particle leaves the sphere at once, and that if n is less than $3/2$, it leaves the sphere at a level which is $2n/3$ above the center.

40. If a particle is placed at any point on the upper half of a smooth sphere, through what arc will it slide before leaving the surface?

41. A particle is constrained to lie on a smooth cycloid with vertex upward, $x=a(2\phi+\sin 2\phi)$, $y=a(\cos 2\phi-1)$. The level of zero speed is na above the vertex. Find the direction and magnitude of the acceleration produced by the constraint of the path at any point (x,y).

42. Two particles are projected from the same point, in the same direction, and with the same speed, but at different instants, in a smooth circular tube whose plane is vertical. Prove that the line joining them always touches a circle whose radical axis with the given circular path is the level of zero speed.

Note. Draw two circles O. O' whose radical axis is horizontal; draw the tangent to O' at any point T, and let it intersect O in P, Q, and the radical axis in R. Draw the perpendiculars PL, QM, to the radical axis, and show that $TP^2:TQ^2=PL:QM$. Also, as T moves on O', show by successive positions that the speeds of PQ are as $TP:TQ$. Hence show that if P moves with the speed due to the zero level MLR, Q does the same.

43. A particle hangs freely from a fixed point by an inextensible string 2 feet in length. It is projected in a horizontal direction with a speed of 20 feet per second. Show that the accelerations produced by the string at the highest and lowest points are as 29 to 5.

44. Given a smooth parabolic tube, convex side up and axis vertical ($x^2=-4ay$). A particle is started from the vertex in this tube with given speed. Show that if the level (h) of zero speed of the particle is below the directrix, the pressure of the tube is always *outwards,* so that the outer half may be removed without affecting the motion; that if this zero level is above the directrix the pressure is always *inwards,* so that the inner half may be removed without affecting the motion; and that if the directrix is the zero level, the pressure of the tube is always zero, so that the particle will describe the path freely.

Note. The normal acceleration of gravity at (x,y) is gdx/ds *inwards,* and tan $\phi=-dy/dx$. Prove the tube exerts the *inward* normal acceleration $\sqrt{ag}(h-a)/\sqrt{(a-y)^3}$, at the point (x,y), etc.

45. A heavy particle hangs freely from a fixed point by a fine elastic wire. The wire produces an acceleration upward proportional to its extension of length and the extension for which the particle hangs at rest is l units. If the particle be pulled down a units, and released, show that it describes harmonic motion of amplitude a, and that its period is the same as a simple pendulum of length l units.

46. A horizontal platform vibrates up and down harmonically in a period of one second. Find its greatest amplitude in centimeters such that particles may rest upon it undisturbed.

CENTRAL FIELDS

47. A partical starts from rest at a given point in a central field whose acceleration towards the center is proportional to the distance from the center. Show that its path is a straight line through the center, and that it executes harmonic vibration about the center whose period is independent of the starting position, and inversely as the square root of the *absolute acceleration of the field.*

48. If a small hole were cut through the earth along a diameter we should have the field in the previous example. Find the time for a particle dropped in at one end to emerge at the other, the diameter being 8,000 miles. Also the velocity with which it reaches the center.

49. A particle is placed on the line joining two centers, the acceleration towards each being proportional to the distance of the particle. Find the center of the motion, and the time of vibration, in terms of the absolute accelerations.

50. If a particle be acted on by repulsive acceleration from a fixed center, proportional to its distance from the center, and start with velocity in a line with the center, show that its motion is in a straight line, and hyperbolic.

51. The earth produces an acceleration towards its center which varies inversely as the square of the distance from the center. A particle is projected vertically from the surface with speed V. To what height will it rise? Compare with the height if the acceleration were constant. Find the least value of V such that the particle never returns to the earth.

52. A particle moves on a given curve in the earth's field. Its arc distance from a fixed point being s, and its distance from the center being r, at time t, show that the tangential acceleration

is $f = -u\,dr/ds$, where $u = a^2g/r^2$, a is the radius of the earth, and g, its acceleration at the surface. Hence from (4), if the distance of the particle from the center is h, and its speed V, at the start, then at time t, $v^2 = V^2 + 2a^2g(r^{-1} - h^{-1})$.

53. Show that a particle moving in a free or constrained path in the earth's field has a speed at any point, the same as if it had fallen in a line with the center to that point, from a spherical surface concentric with the earth, which is the surface of zero speed for the moving particle.

Note. Let k be the distance from the center, in falling from which to its starting position, the particle would have acquired its initial speed V. By the preceding example, $V^2 = 2a^2g(h^{-1} - k^{-1})$. Hence find v^2, etc.

54. Extend the results of the two preceding examples to a central field which produces on a particle at distance r, an acceleration $F'r$, (away from the center positive, and towards the center negative).

Ans. $f = F'r.dr/ds$, $v^2 = 2(Fr - Fk)$, where Fr is a function of r whose differential is $F'r.dr$. Draw the spherical locus of points at distance k from the center of the field, and the radius r of the particle cuts it in the position by falling from which to its given position the particle would acquire its given speed v.

55. A particle P is on the inner surface of a smooth sphere of center C, in a field of center O, whose acceleration G on P, varies only when the distance OP varies. Draw the small circle of the sphere through P whose center L is on the line OC, and show that if G and CL are alike in sign (*negative towards* O), then the particle will describe this circle freely, and with uniform motion, if started upon it with speed v given by

$$v^2 = (OL/OP).(LP)^2.G/CL.$$

RESISTANCE OF A MEDIUM

56. The free path of a particle in a resisting medium alone is a straight line. In a constrained path, the path produces the whole normal acceleration v^2/R.

57. If the retardation vary as the speed, speed is destroyed in proportion to the space described. The motion is geometric towards a limiting position of rest as center.

58. If the retardation vary as the square of the speed, the reciprocal of the speed increases uniformly at a rate measured by the *absolute retardation* of the medium. The speeds at successive equidistant points on the path are in decreasing geometric progression. The distance passed over is ultimately indefinitely large, with no limiting position of rest.

59. Find the time in which a particle with given speed will

be reduced to rest, the medium retarding as the square root of the speed.

60. Consider a retardation varying as $v(a+v)$.

61. Consider a retardation varying as $v(a^2+v^2)$.

THE CONSTANT FIELD AND A MEDIUM

62. Show by $dv=fdt$, that the speed of a falling particle must continually approach the speed at which the retardation of the medium balances gravity, as a *terminal* value, decreasing if greater than, increasing if less than, and uniform if equal to such terminal value.

63. Let the medium retard as the speed, and denote the absolute retardation by k. Consider a particle in vertical motion (rising or falling), starting with speed V, and passing over a distance s in t units of time.

(a) When rising.

$dv=gdt-kds$, and $v=V-gt-ks$

$dt=-dv/(g+kv)$, and $kt=log.[(g+kV)/(g+kv)]$

(b) When falling,

$dv=gdt-kds$, and $v=V+gt-ks$,

$dt=dv/(g-kv)$, and $kt=log[(g-kV)/(g-kv)]$

Note. The two cases differ only in the sign of g. Going up, g is retardation, or $-g$ is acceleration; going down, g is acceleration; k is retardation in either case.

(c) Show from the preceding equations that the falling particle is approaching the terminal speed g/k.

(d) In vertical motion, the speed produced by gravity in any time is diminished by the medium in proportion to the space described during that time.

(e) A particle is projected vertically upwards with speed V; find its greatest height H, and time T of reaching it; also the time T' of return to the point of projection, and the speed V' of return.

Ans. $kT=\log(1+kV/g)$, $kH=V-gT$; and by solution of $kT'=-\log(1-kV'/g)$, and $kH=gT'-V'$.

(f) A particle is projected vertically with a speed of 96 feet per second, and returns to the point of projection in 5 seconds. Find: the return speed, the absolute retardation of the medium, the times of rising and falling, and the highest point reached.

Ans. 64, .246, 2.247, 2.753, 97.94.

Note. Take $g=32$, and there are six unknowns in the four equations of (e) above. The two additional equations, $V=96$, $T+T'=5$, are therefore sufficient to determine them all. V' is at

once found by subtracting the two values of H. Adding the values of T, T', then gives, $5k—\log[(1+3k)/(1—2k)]=0$.

Solution is by successive trials. Thus, the first member is positive for $k=.2$, and negative for $k=.3$; hence it is zero for some value of k between .2 and .3; etc. Multiplying the equation by .4343 reduces the natural to a common logarithms, if no table of natural logarithms is to be had.

(g) A particle is $2\frac{1}{4}$ seconds in rising, and $2\frac{3}{4}$ seconds in returning to the point of projection. Determine the absolute retardation, etc.

(h) A particle is projected upward with its terminal speed for the medium. Determine the height of ascent, time of ascent and descent, and speed of return to the point of projection.

(i) One particle begins to fall from the higher extremity of a vertical line, at the same instant in which another is projected upward from the lower extremity with given speed. Find the time in which they will meet.

64. Work out the cases of the preceding example when the retardation of the medium varies as the square of the speed. Let $a=g^{\frac{1}{2}}/k^{\frac{1}{2}}$, the terminal speed.

65. *The Motions of type (f) Further Considered.* Show the effect of a medium retarding as the speed, on the vibrations of a simple pendulum, and on all motions that would be harmonic in vacuo.

Note. With the center of harmonic motion as origin, the tangential acceleration in vacuo is expressed by $—m^2s$. The medium contributes the additional amount $—kDs$, by the law stated, since Ds is the speed. Hence, in the medium

$$D^2s+kDs+m^2s=0.$$

By substituting $s=x.\exp.(—\frac{1}{2}kt)$ this reduces to

$$D^2x+(m^2—\frac{1}{4}k^2)x=0.$$

This shows that x determines a harmonic motion, or a hyperbolic motion, or a uniform motion, according as $m^2—\frac{1}{4}k^2$ is positive, negative or zero. Also, the factor, $\exp(—\frac{1}{2}kt)$ decreases geometrically with the time.

66. The amplitude of a seconds pendulum is reduced from 3 to 2.6 inches in one period. Find the absolute retardation of the medium, and the period in vacuo.

67. Two particles, connected by a fine elastic string, are falling, with the string vertical, and extended to its natural length. The upper particle is suddenly stopped; find the subsequent motion of the lower particle, the air retarding as the speed.

Ans. If n^2 be the absolute tension, k the absolute retardation, na the speed when the upper particle is stopped,

$s=a\,exp(—\frac{1}{2}kt)\sin nt$.

OBLIQUE AND RECTANGULAR COORDINATES

Choose any convenient axes OX, OY, OZ, in space, and let (x,y,z) be the coordinates of the moving point in its position P, at time t, and (x',y',z') its coordinates in the position P' at time t'. Thus, x,y,z, will be given functions of t, and x',y',z', will be the same functions of t'.

If we make the velocity PS the diagonal of a parallelopiped whose edges are parallel to the axes, those edges, from P as initial point, are called the *components* of PS, the one parallel to OX being *the x-component*, etc. These components are expressed by the positive or negative numbers which measure them (positive when in the directions of the axes). Conversely, given the components of a velocity, construct them in their lengths and directions parallel to the axes, from the point of application P, and complete the parallelopiped with the constructed lines as edges; then its diagonal, PS, represents the velocity determined by them.

With rectangular axes *the magnitude of the velocity is the square root of the sum of the squares of its components.*

The x-component of the resultant of two velocities is the sum of the x-components of the velocities, and similarly for the other components.

If two velocities are parallel, then corresponding components are proportional, and in the ratio of the velocities (opposite velocities in a negative ratio).

These are simply geometrical laws of displacements, and hold for all quantities which are properly represented by displacements, as velocities, accelerations, etc. A line representing any such quantities is called a *vector*.

We proceed to find the components of the velocity and acceleration of a moving point, in terms of the functions, x, y, z, which fix its position at any time t, from their definitions as limits of average values.

Since $[x,y,z]$ are the components of the displacement OP, and $[x',y',z']$, of OP', the components of PP' will be $[x'-x, y'-y, z'-z]$, and the components of average velocity PS', if $1/(t'-t)=n$, *will be* $[n(x'-x), n(y'-y), n(z'-z)]$. *Hence*:

(5) *The components of the velocity at time t are* $[Dx, Dy, Dz]$.

These components are functions of t which we denote for the moment by $[u,v,w]$, and by $[u',v',w']$, at time t', (respectively the components of OH, OH', fig. 1). Thus, the components of HH' are $[u'-u, v'-v, w'-w]$, and those of the average acceleration HT', *are* $[n(u'-u), n(v'-v), n(w'-w)]$, where $n=1/(t'-t)$; and consequently their limits are $[Du,Dv,Dw]$ the components

of the acceleration HT. Hence, replacing u, v, w, by their values, Dx, Dy, Dz:

(6) *The components of acceleration at time t are, D^2x, D^2y, D^2z.*

AREAL SPEED AND VELOCITY

We consider the conical area generated by the "radius vector" of the moving point, and called the radius vector area. If U *denote* that area OAP, at time t (fig. 2), which becomes U'=area OAP' at time t', then $U'—U$ is *the areal increment* in time $t'—t$; $(U'—U)/(t'—t)$ is the average *areal speed;* and its limit DU, is the *areal speed,* at time t.

On the other hand, consider *the areal displacement* represented by *the plane triangle OPP'*, and call $OPP'/(t'—t)=OPS'$, *the average areal velocity,* from time t to time t', and its limit, OPS, *the areal velocity* at time t. Since the conical and plane areas OPP', OPP', have a limiting ratio of unity, as P' approaches coincidence with P, the areal velocity and speed have the same magnitude. Also, the areal velocity OPS, is a tangent plane to the conical surface, along the element OP.

COMPONENTS OF AREAL VELOCITY (Rectangular Axes)

The xy-component of the areal velocity OPS, is its projection on the *xy plane* by lines *parallel to OZ*. Since (x, y, z) and $(x+Dx, y+Dy, z+Dz)$ are the coordinates of P, S, the coordinates of their projections P'', S'', are (x,y) and $(x+Dx, y+Dy)$; and the area of the projection $OP''S''$ is consequently by plane analytics, $\frac{1}{2}(xDy—yDx)$. With oblique axes, the additional factor sinXOY is required. Similarly for the yz and zx projections. Hence,

(7) $\frac{1}{2}(yDz—zDy)$, $\frac{1}{2}(zDx—xDz)$, $\frac{1}{2}(xDy—yDx)$

are the yz, zx, xy, components of areal velocity at time t.

By the differentiation D, of these components, we obtain the components of *areal acceleration,*

(8) $\frac{1}{2}(yD^2z—zD^2y)$, etc.

MOMENTS ABOUT A POINT.

Consider any vector PS, whose components are (X, Y, Z), and point of application, $P=(x,y,z)$. Then, *the moment of PS about any point O, is double the triangle OPS,* considered as to plane *OPS direction PS about O,* and *magnitude,* the area $2OPS$. Consequently, the moment is not changed by transfer of P to any point of the line of action, PS. Its components are defined and found as in areal velocities, and are (in rectangular axes)

$yZ—zY, zX—xZ, xY—yX$, on the yz, zx, xy, planes.

In particular, *the moment of velocity is double the areal velocity,* and *the moment of acceleration is double the areal acceleration.*

The resultant of two moments $2OPS, 2OPS'$ is defined as *the moment $2OPR$,* where PR is the resultant of PS and PS'

The components of the resultant of two moments are the sums of the corresponding components of the moments.

For, $y(Z+Z')-z(Y+Y')=(yZ-zY)+(yZ'-zY')$ etc.

MOMENTS ABOUT AN AXIS

The moment of a vector PS about an axis OZ, is defined as *the orthogonal projection of its moment $2OPS$ on a plane perpendicular to OZ.*

Thus with rectangular axes, the yz, zx, and xy components of the moment of PS about O, are also the moments of PS about the axes of $x, y,$ and z, respectively.

With rectangular axes, construct the line OL whose axial components are the moments of PS about the axes, area measure in the latter being taken as length measure in the former, and yz components of area as xcomponent of length, etc.

In rectangular axes, the sum of the squares of the components of an area equals the square of the area. Since the length of the line is found in the same way from its components, therefore: *The measures of the length OL, and the area $2OPS$, are equal,* i. e., the magnitude of OL represents the magnitude of the moment $2OPS$.

Also, with rectangular axes, two lines are perpendicular when the sum of the products of corresponding components is zero. Hence show that OP, whose components are (x,y,z), and PS, whose components are (X,Y,Z), are each perpendicular to OL, whose components are ($yZ-zY$, etc.) Thus, *OL is perpendicular to the plane, OPS, of the moment.* It will be named *the axis of the moment.*

Finally, *the angular direction of a moment, clockwise or counter-clockwise, as seen from the end of its axis is the same for all moments.* For, conceive P and S to move continuously in any way so that the angle POS does not vanish; then the components of the moment $2OPS$, change continuously. Since these components and the components of the axis OL, have the same measues, therefore L also moves continuously. Remembering further, that OL always remains perpendicular to the plane OPS as the motion of P and S continues, and that the angle POS does not vanish, so that OL does not vanish, it is plain that the aspect of the angle POS from L cannot change, but remains permanently clockwise, or counter-clockwise, in all positions. By taking $(1,0,0)$, $(0,1,0)$ for P and S, we find $L=(0,0,1)$. Thus *the*

aspect of any moment from the end of its axis is the aspect of the angle XOY from a point on OZ. The axes are said to be in *right-hand order,* or *left-hand order, in space,* according as this aspect is *counter-clockwise* or *clockwise.*

The axis OL of a moment $2OPS$, determines it in all essential particulars, *magnitude, plane,* and *angular direction.* Further, considering moments about one point O, *the axis of the resultant of two moments is the vector resultant of their axes.* For the area components of a moment, and the length components of its axis, are identical measures in rectangular coordinates, and resultants of moments are found by adding corresponding components, as are also the resultants of vectors.

The axes of moments are therefore vectors, and are called *moment-vectors.* Areal velocities and accelerations are similarly represented by vectors, whose components in rectangular axes are in length what the area components are in area. Their x-components are the yz-components of the areas, etc., they are perpendicular to the planes of the areas and represent their magnitudes, and their angular directions as in moments.

EXAMPLES II

1. The differential and integral equations of uniform motion in a straight line are

$$D^2[x, y, z]=[O, O, O];D[x, y, z]=[a, a', a''].$$
$$(x, y, z)=(at+b, a't+b', a''t+b'')$$

2. The equations of free motion in a constant field are,

$$D^2[x, y, z]=[a, a', a'']; \; D[x, y, z]=[at+b, a't+b', a''t+b''];x=\tfrac{1}{2}at^2+bt+c, \text{ etc.}$$

3. Show that the differential equations of free motion in a central field, with the center as origin, are:

$$D^2[x, y, z]=(F/r)\,[x, y, z]$$

where r is the distance from the center to the point (x, y, z), and F is the *repulsive* acceleration of the center (a function of $x, y,$ and z, which must be negative at points where the center produces *attractive* acceleration).

Note. The acceleration at P is parallel to OP, since O is the center; and its ratio to OP is as F to r, where F is negative if the acceleration is towards O. Also, corresponding components of parallel vectors are in the ratio of the vectors.

5. Find the coordinate equations which determine the path in ex. 1. Find the areal velocity, and show from it that the radius vector of the moving point describes equal areas in equal times in a fixed plane. What does it signify when the vector of areal velocity is parallel to a fixed line?

6. With the origin at the starting position of the particle in ex. 2, find the areal velocity, and show thence that the path is a plane curve, and that the area described by the radius vector varies as the cube of the time.

7. Show that the free path of a particle in a constant field lies in the fixed plane which contains the velocity and acceleration at the beginning of the motion.

Note. Take the fixed plane as XOY, and show successively, that D^2z, Dz, z are each zero at any time t. Or, conceive the particle constrained by contiguous planes to lie in the plane named, and show that these planes produce no accelerative constraint at any time, and may thus be removed without affecting the motion. Or, find the vector areal velocity from the general equations (ex 2) and show that it is parallel to a fixed line.

8. If we take the axis OX in the direction of the initial velocity (a), and the axis OY in the direction of the acceleration (g) of a constant field, find the differential equations of free motion, and thence find $x=\frac{1}{2}gt^2, y=at$. By eliminating t we find the equation of the path, $y^2=2a^2x/g$. Interpret this motion as the resultant of uniform motion in a straight line, and uniformly increasing vertical motion. Also show that the path is a parabola, with axis downward, and that the speed of the particle at any time is the speed it would acquire by falling to its position from the directrix.

Note. By analytical geometry, $y^2=4a'x$ is the equation of a parabola, whose axis is in the direction OX, with the origin on the curve at distance a' from focus and directrix and with the axis OY a tangent.

9. A particle is started with a speed of 40 feet per second, at a slope of 3/4 to the vertical. Find the directrix and focus of the path. Construct the flight of the particle at intervals of one-fourth of a second for the first two seconds. Draw the hodograph of the flight, showing the velocities at the same time.

10. The same as ex 9, the slope to the vertical being 4/3.

11. Plot the points of the speed curves of the preceding motions at the times named; also find their equations.

12. Find the horizontal range for a particle projected with given speed a, at an angle of depression p from the vertical and the depression for maximum range.

Note. The time of flight is found by the condition that in that time the height to which the initial velocity has taken the particle is equal to the distance fallen by gravitation.

For answer take θ a right angle in the next problem.

13. In the preceding example, find the range on a plane of depression θ from the vertical and the maximum range.

Ans. $r=2a^2\sin(p-\theta)\sin p/g\sin^2\theta$.

For maximum range $p=\frac{1}{2}\theta$, $r=a^2/g(1+\cos\theta)$, the *polar* equation of the curve which bounds the greatest distance that can be reached in any direction with initial speed a, (a parabola with the starting point as focus, axis downward, and highest point the height of vertical flight). It is the *envelope* of the paths of all particles starting from the origin with the same speed in the same vertical plane.

15. A buys a flywheel of B 16 feet in diameter, with the guarantee that it will run safely 80 revolutions per minute. The wheel burst and a piece was found at a distance of 1,000 feet. A sues B for damages; should he recover?

16. Show that for a free path in a central field, with the center as origin, the areal acceleration is *zero*, the areal velocity, *constant;* and that consequently the free path of a particle in a central field lies in a fixed plane containing the center, and the radius vector from the center to the moving point describes equal areas in equal times. Show also the converse, that if any free path be a plane curve through a fixed point, and the radius from that point to the particle describes equal areas in equal times, then the field is central, the fixed point the center.

17. For a free path in any field show that the radial acceleration at time t is, in rectangular coordinates, $D^2x.(x/r)+D^2y.(y/r)+D^2z.(z/r)$.

Note. The *radial acceleration* is the orthogonal projection of the acceleration upon the radius vector OP, and is therefore the sum of the orthogonal projections of its axial components. The other component of acceleration, perpendicular to the radius vector, is called the *transverse acceleration.* Its component on the x-axis is found by subtracting the x-component of radial acceleration from D^2x, and is $2(Y''z-Z''y)/r^2$, where X'', Y'', Z'', are the components of areal acceleration.

18. Kepler announced, as the result of astronomical observations, covering many years, that *the path of a planet is an ellipse, with the sun at a focus; that the radius vector from the sun to a planet describes equal areas in equal times; and that for different planets, the cubes of the major radii of their orbits are as the squares of their times of revolution.* Newton thence proved: (a) *the acceleration on a planet forms a central field with the sun as center;* (b) *towards the sun and inversely as the square of the distance;* (c) *absolute acceleration the same for all planets.*. Prove the same in order, (a) from the plane path and law of areal speed,

(b) from the form of the path and situation of the sun, (c) from the relation between periodic times and radii of orbits.

Note. Newton did not announce the universal law of gravitation at that time, because, owing to erroneous data regarding the earth, it appeared to fail in the case of the motion of the moon. When eighteen years later more accurate data verified the law, he immediately made the announcement.

Outline of Proof. (a) Since the plane and magnitude of the areal velocity are constant, its components are constant, and the components of areal acceleration are zero. Thus, with the sun as origin, the components of acceleration (D^2x, etc.) are proportional to (x, y, z) (*Q.E.D.*)

(b) Take the plane of the orbit as the plane of xy (fig. 1), the major axis as x-axis, and the perpendicular line through the focus F occupied by the sun, as axis of y; $P=(x, y)$ is the position of the planet at time t, and FP is r. Let h be the area described by FP in unit time. Then, e being the excentricity of the orbit, and $2l$ the latus rectum,

(1) $x^2+y^2=rr$. (2) $r=l-ex$, by geometry of the ellipse, and from areas, (3) $xDy-yDx=2h$.

Differentiate (1), substitute the value of Dr from (2), and solve for Dx, Dy, with (3), making use of (1) and (2) in the result. Thus, if $c=2h/l$ (4) $Dx=-cy/r$, (5) $Dy=c(e+x/r)$.

Now differentiate (5), and after reduction by preceding equations, we find $F=rD^2y/y=-2ch/r^2$, *Q.E.D.*

(c) Let the major and minor axes be $2a, 2b$. The positive value of F when $r=1$, or the absolute acceleration of the sun's field on the given planet P, is $k=2ch$, *or* $4a(h/b)^2$, *since* $l=b^2/a$; and the time of revolution is $T=\pi ab/h$. Eliminating h/b, $k=4\pi^2a^3/T^2$, which is the same for all planets.

19. Show that the hodograph of the motion of a planet is a circle.

Note. Take axes through O (fig. 1) parallel to the preceding axes through F. The coordinates of the point H corresponding to the planet P are then $x'=Dx, y'=Dy$. Solve (4) and (5) above for $x/r, y/r$, and substitute in (1). We thus obtain,

$x'^2+(y'-ec)^2=cc$, *a* circle, *center* $(0, ec)$, radius, c.

20. If the path of a comet is a parabola or hyperbola, with the sun as focus, and the radius vector from the sun to the comet describes equal areas in equal times, determine the law of acceleration, and the hodograph.

Note. The work is the same as (a) (b) above, with e equal to or greater than unity. The hodograph is a circle, with O on or outside its circumference. If the hyperbolic path is the branch about the empty focus, then, $r=ex-l$, which is equivalent to

changing the signs of e and l, and therefore of c, in the above work. The field is then *repulsive* and inversely as the square of the distance, and the hodograph is a circle.

21. A planet whose radius vector from the sun is d, is projected at right angles to that radius with speed V; determine the excentricity of its orbit, its major and minor axes, and the period of revolution.

Note. $Vd=2h$, from the areal speed, $V=c(1+e)$ from the hodograph, and k is $2ch$.

Ans. $e+1=V^2d/k, a=kd/(2k-V^2d)$, etc.

22. If the above orbit is a circle, find the initial velocity, center, and character of the motion. When is the orbit a parabola? an hyperbola?

23. A particle is started horizontally with speed just sufficient to make it go round the earth in vacuo; find the time of revolution.

24. Find the absolute acceleration such that a particle will describe a circular orbit of unit radius in one unit of time. Find the time of revolution if the absolute acceleration is unity.

25. Find the velocity with which a particle must be started in any central field whose acceleration varies as a function of r, in order that its orbit may be a circle, and determine the character of the motion, and the time of revolution.

26. Given that the moon revolves in a circular path about the earth in 28 days, find its distance from the center of the earth. Ans. Between 29 and 30 diameters of the earth.

27. A fine elastic string, fastened at one end, and just reaching without stretching, a smooth *fixed* ring G, is passed through the ring and attached to a particle. The constant field acting, the particle is found to rest in equilibrium at a point O below G; it is then drawn aside to a given point A, and projected with a given velocity AC. To find its position P at any time t, its path, and the character of its motion.

Note. Let c be the absolute acceleration of the string on the particle; then at O, its acceleration is $c.OG$, and $c.GO$ is the acceleration of the field, since there is equilibrium at O. The accelerations on the particle in any position P, are therefore, $c.PG$, by the string, and $c.GO$, by the field, whose resultant is $c.PO$. Thus, the resultant field is central, towards the center O, and its acceleration is cr at distance r. Consequently (ex. 16) the motion of the particle is in a fixed plane through O, which must be the plane OAC containing the initial velocity. Draw OB parallel to AC; and take OA, OB as axes of x and y, and $P=(x, y)$. Hence (ex3), $D^2x=-cx, D^2y=-cy$.

If $OA=a, AC=b\sqrt{c}$, the solution is

$x=a\cos(\sqrt{c}.t), y=b\sin(\sqrt{c}.t)$,

and for the path, $(x/a)^2+(y/b)^2=1$, the equation of an ellipse referred to semi-conjugate diameters $OA=a, OB=b$.

In another plane through OA take OB' perpendicular to OA as a new axis of y'; and consider the point P', whose coordinates are $x=a\cos(\sqrt{c}.t), y'=a\sin(\sqrt{c}.t)$. Show that as t varies P' describes a circle AB', uniformly c radians per unit time, as P describes its elliptic path AB, and that $P'P$ is always parallel to the fixed line $B'B$. The motion of the particle is therefore, *the parallel projection of uniform circular motion whose period of revolution is* $2\pi/\sqrt{c}$. This motion of P is called *elliptic harmonic motion;* it is not, however, a harmonic motion on the ellipse. Harmonic motion in a straight line is called *simple harmonic motion.*

28. A particle is projected from the origin with an initial velocity whose components are a horizontal, and b vertical, and the same component are u, v, at time t, when the particle is at (x, y). The retardation of the air varies as the speed (its absolute value being k), and gravity also acts. Find equations for u, v, x, y, in terms of t. Also equations for the horizontal range and time of flight, and the equations for maximum range, when the initial speed is given.

RESULTANT MOTIONS

The general definition of the resultant of two motions is that, *if P. Q. be the moving points of the component motions at any time, and O is a fixed point, then completing the parallelogram OPRQ, R is the moving point of the resultant motion as to the origin O.* Conceive a space carried by the moving point P, but *with directions unchanged*; the point R is then carried with P in this space, and has besides, the motion round P which Q has round O. The motion of Q round O is called the *relative motion of R as to P,* i. e., OQ=PR in length and direction at every instant of time.

The velocity and acceleration of the resultant point R above are the resultants of the velocities and accelerations of P and Q, since, as displacements, $OR=OP+OQ$, and hence for velocities, $D.OR=D.OP+D.OQ$, etc.

EXAMPLES III

1. Show that a translation of the origin O to a new position O' translates the resultant of two given motions the opposite amount, $O'O$.

2. Show that the parallel translation of a component motion affects the resultant motion by the same translation.

3. The resultant of two simple harmonic motions of the

same period and direction is a simple harmonic motion. Also find the resultant amplitude and phase.

4. The resultant of two or more simple harmonic motions of the same period and phase, on different straight lines, is simple harmonic motion of the same period and phase.

5. The resultant of two or more harmonic motions of the same period on different lines is elliptic harmonic motion (becoming in special cases uniform circular or simple harmonic motion).

6. Show that a simple harmonic motion is the resultant of two uniform circular motions of opposite directions.

7. Show that the resultant of two simple harmonic motions in rectangular directions, of equal phase, and periods as 1:2, is a parabolic vibration.

Note. $x=a\cos 2nt,\ y=b\cos nt,$ and eliminating $t,$

$$(x/a)+1=2(y/b)^2.$$

8. In the same case, if there is a difference of phase, the resultant path is a figure 8 curve.

9. Construct the path of the motion

$$(x, y, z)=(4\cos nt,\ 4\sin nt,\ 3\cos 2nt).$$

Note. Take a sheet of paper, 8 units by 8π units and with the lengthwise middle line as axis of x, and an edge perpendicular to it as axis of y, construct the harmonic curve $y=3\cos\frac{1}{2}x$, from x zero to its extreme value, 8π. Fold the paper into a cylinder of radius 4 units, and the curve upon it takes the form of the required path.

10. If the surface of the above cylinder be turned about its axis (the axis of z) through the angle $-b$, the axes of x, y, z, not moving, the curve drawn becomes the path of the motion,

$$(x, y, z)=[4\cos(nt+b,\ 4\sin(nt+b),3\cos 2nt]$$

11. Show that the orthogonal projections of the motions in exs. 9 and 10 on the plane of xz are the motions given by the x and z coordinates alone.

12. Find the resultant of two uniform circular motions of the same period and plane. When is their resultant a simple harmonic motion?

13. *The curve of pursuit.* A point in a plane moves uniformly in a straight line (the axis of x). A second point moves in the plane so that its direction of motion is always towards the first point. Assuming that its line of motion in the beginning is the axis of y, find the differential and integral equations of its subsequent motion.

14. A hound sights a fox 100 yards due south. The fox runs due east, and is overtaken in 300 yards. Show that the hounds runs 354.14 yards.

Note. If the hound run n times as fast as the fox, and r be the distance between them at time t, show that $ds+dr=dy/n$, integrate, and solve for n.

PLANE POLAR COORDINATES

Consider a point moving on a plane curve APP' (fig. 3), whose position is P at time t, etc. Let O be the pole, and OX, the the initial line, and with$O$ as center, draw a circle of *unit radius*, cutting OX at L, OP at Q, etc. Then the polar coordinates of P are $\theta=$arc LQ, and $r=OP/OQ$. These are the coordinates of the moving point at time t, and are functions of t, which become $\theta/=$arc LQ', $r'=OP'/OQ'$ at time t'.

Let QR be a positive quadrant on the unit circle. Then OQ, OR, are respectively *radial* and *transverse* directions at time t, and all vectors at that instant, may be expressed by their components in these directions, called their *radial* and *transverse* components. Thus, the radial and transverse components of the velocity, PT, at time t, are PS and ST, where PS is a continuation of OP, and OST is a right angle. We proceed to determine the radial and transverse components of velocity and acceleration.

The average velocity from time t to time t' is represented by the vector nPP' (using n for $1/(t'-t)$ here and in what immediately follows). In the vector addition and substraction, nPP' is the same as $n(OP'-OP)$. or $n(r'OQ'-rOQ)$. Combining with this the null system, $-nrOQ'$, $+nrOQ'$, and noting that $OQ'-OQ=QQ'$, we obtain, $nPP'=n(r'-r)OQ'+r.nQQ'$.

The limit of this average velocity is the velocity PT; also, the limit of nQQ' is the velocity of Q on the unit circle, or $D\theta.OR$. Thus, $PT=Dr.OQ+rD\theta.OR$; and since we express the units of direction, OQ, OR, by the terms *radial* and *transverse*, we have,

(9) $(Dr, rD\theta)$ *are the radial and transverse components of velocity.*

Note. If u, v, stand for the units OQ, OR, the preceding work is the differentiation of the vector displacement $OP=ru$, *viz.*,

$D(ru)=Dr.u+rDu=Dr.u+rD\theta.v$ (since Du is seen to be $D\theta.v$)

The acceleration is found by a similar differentiation of the velocity, allowing for the variation of both units, u, v, in the above note; also, observe that Dv is the velocity of R, or $-D\theta.u$ Thus let q stand for $rD\theta$, and the acceleration is, after collecting coefficients of u. v:

$D(Dr.u+q.v)=(D^2r-qD\theta)u+(Dr.D\theta+Dq)v$.

If we multiply the coefficient of v by r, it becomes $Dr.q+rDq$, i. e., $D(rq)$. Thus finally,

(10) $F = D^2r - r(D\theta)^2$, *is the radial component of acceleration.*

(11) $G = r^{-1}D(r^2D\theta)$ *is the transverse component of acceleration.*

The triangle OPT is the areal velocity, and if U denote the radius vector area OAP, v the speed Ds, and p the perpendicular ON, on PT, the magnitude of this area, or the areal speed is expressed in the forms,

(12) $DU = \frac{1}{2}r^2D\theta = \frac{1}{2}pv$.

Note. v is found in polar coordinates by the right triangle PST.

(13) $vdv = fds = Fdr + Grd\theta$ by projecting F and G upon PT, to find f.

Polar coordinates are particularly useful when considering plane motions in central fields. Since the whole acceleration is radial, this gives at once $G = O$, i. e., $D(r^2D\theta) = 0$. Hence,

(14) $r^2D\theta = pv = H$, a constant, (in a central field).

This expresses, of course, that the radius vector describes equal areas in equal times, as heretofore shown $[DU = \frac{1}{2}H,\ U = \frac{1}{2}Ht + c]$. By this equation, which is $dt = r^2d\theta/H$, we may eliminate dt from the differential equations.. Let a small cap D stand for $d/d\theta$; Then D become everywhere $(H/r^2)\mathrm{D}$.

i. e., $Dr = (H/r^2)\mathrm{D}r = -H\mathrm{D}(1/r)$, etc. The most important equations thus holding for a central field, with r and θ the only variables involved are,

(10') $F = -(Hu)^2(\mathrm{D}^2u + u)$, ($u$ standing for $1/r$)

(12') $pv = H$, whence $vdv = (H/p)d(H/p)$.

(13') $vdv = fds = Fdr$.

EXAMPLES IV

1. Find the laws of the central fields in which a particle will describe the following curves, and determine the starting conditions that it may do so, in particular the curve of zero speed.

(a) The hyperbolic spiral, $r\theta = a$.

(b) The equiangular spiral $p = r \sin c$. [(12') and (13')]

(c) The logarithmic spiral $\log (r/a) = \theta/c$. [The same as (b)].

(d) The spiral in which r/a is the n'th power of sec (θ/n).

(e) $r = a \sec n\theta$, and $r = a \operatorname{sech} n\theta$.

(f) The spiral of Archimedes, $r = a\theta$, and the rose, $r = a\sin 2\theta$

(f) The cissoid $r = a\sin\theta\tan\theta$; the lemniscate $(r/a)^2 = \cos 2\theta$.

(g) A circle about a point on its circumference; a parabola about its vertex; about the intersection of axis and directrix.

(h) A conic section about a focus, $l/r=1+e\cos\theta$.

(i) An ellipse or hyperbola about its center.

(j) An equilateral hyperbola about its center, $(r/a)^2=\sec 2\theta$.

2. A particle describes an ellipse, its acceleration being always perpendicular to one axis; find the law of acceleration.

3. What would be the path of a projectile if gravity varied inversely as the cube of the distance from a horizontal plane?

4. If the acceleration varied as the distance in the preceding example?

5. Determine the path when the law is inversely as the square of the distance.

6. Consider the laws inversely as the cube of the distance, and inversely as the distance.

7. If the hodograph is a circle, and the field central, the law must be inversely as the square of the distance.

8. The speed varies inversely as the distance from the center; determine the law of the field, and the path.

9. A planet is projected with the speed for a circular path at its given distance, but at an angle of 45 degrees to its radius vector. Show that the point of projection is the end of a minor axis of its orbit, and its axes are as $\sqrt{2}$ to 1.

10. A particle, acted upon by two centers, moves with constant speed, and the product of its distances from the centers is constant. If one center vary as the distance show that the other varies as its inverse cube.

11. Show that the radius of the sphere of zero speed for a planet is twice the major radius of its orbit, or that $\frac{1}{2}v^2=k/r-k/2a$

12. Consider the motion of a particle in a field of two centers, each varying as the distance.

13. A particle moves in one plane, and its acceleration is constant in magnitude, but revolves uniformly in direction. Find the motion and path described, starting from rest.

Ans. With the x as the initial position and the x axis as the initial direction of acceleration; n, the angular speed of the latter, and an^2, its magnitude, the motion is $x=a\,\text{vers}\,nt$, $y=a(nt-\sin nt)$, the motion of a point on a circumference of radius a, rolling uniformly on the y axis.

14. A particle describes the cycloid $x=a\,\text{vers}\,\phi$, $y=a(\phi+\sin\phi)$, in a fiield whose acceleration is always parallel to the base; determine its law.

Ans. The base is the line $x=2a$. If b is the constant ve-

locity parallel to the axis, the acceleration is $-b^2/a\sin\phi\text{vers}\phi$, in the direction OY.

15. If a particle be acted on by a vertical acceleration so as to describe the common catenary $y=a\cosh(x/a)$, find the acceleration and speed at any point.

Ans. c^2y, cy, where c is the ratio of the constant horizontal velocity to the radius a.

16. A particle is moving in a plane, and its acceleration is always perpendicular to the line joining it to a fixed point of the plane. To find the law of the field in order that the angular velocity of the particle about the point may be constant, and to determine the path.

Ans. $\theta=nt$, and F is zero; whence
$r=a\exp(nt)+b\exp(-nt)$, $G=2nDr$.

17. Show that any number of given centers, whose accelerations vary as the distance, are equivalent to a single center of the same law.

Note. Let l, m, ... be absolute accelerations of centers A. B,.... positive or negative as repulsive or attractive; the resultant on P is $lAP+mBP+\ldots$, and $lAE+mBE+\ldots=0$, where E is the position of equilibrium. Taking this null system from the resultant, it becomes, $(l+m+\ldots)EP$.

18. Show that if the algebraic sum of the absolute acceleration above is zero, the resultant field is constant.

19. An attractive center which produces acceleration in proportion to the distance, moves uniformly in a straight line in a given plane. To determine the motion of a particle in the plane whose initial circumstances are given.

20. Two particles P, P' produce attractive accelerations, each on the other towards itself, and varying as the distance between them. With any initial circumstances, describe their subsequent motion.

Ans. Call the absolute acceleration of each its mass. Then their center of gravity *moves uniformly in a straight line.* Each revolves *in elliptic harmonic motion relative to the other as center with a mass the sum of their masses.*

21. The same as the preceding, except varying inversly as the square of the distance.

Ans. Their center of gravity moves uniformly in a straight line. Each revolves in a conic section relative to the other as focus, with mass the sum of their masses.

22. The same as ex. 20, except varying as any given function of the distance, with different absolute values.

23. Every point of a plane of infinite extent produces ac-

celeration on a particle towards itself, of absolute value *k per unit area,* and inversely as the square of the distance. Determine the field produced.

24. Using ex. 21 to modify Kepler's third law, show that the mass of the sun is approximately 356,420 times the mass of the earth, the sun's distance from us being 400 times the moon's, the earth's mass 75 times the moon's, and its periodic time 13.4 times that of the moon.

25. In an orbit in any central field, draw lines paralell to the tangents at two points of the orbit at distances inward and inversely proportional to the speeds at the points of contact. Show that the line joining the intersection of the tangents to the intersection of their parallels passes through the center of the field (use $pv=H$).

26. If a central orbit pass through the vertices of a triangle, and the velocities at them be parallel and proportional to the opposite sides, prove the center is the center of gravity of the triangle.

THE HALF SQUARE OF SPEED.

It will be convenient here to call the half square of the speed of a particle its VIS VIVA. Vis viva is a term that has been used in various senses by writers on dynamics, any of which make it some multiple of the square of the speed.

If we integrate $vdv=fds$ between given points of a given path, we obtain $\frac{1}{2}v^2-\frac{1}{2}V^2=\int fds$, where V is the beginning, and v, the ending speed, and the integral is taken over the path. In a field, f will be determined at every point of the path, whether the motion over the path be free or constrained, whatever the initial speed may have been, and whenever the path be described; i. e., f is a function of s as a geometric coordinate of the path. Hence,

The increase of vis viva produced by a field depends only upon the path described.

A field is called *a conservative field* if, still further, the increase of vis viva depend only upon *the beginning* and *ending* points, by whatever path motion takes place from one to the other.

If (X, Y, Z) be the components of the field at the point (x, y, z), then Xdx/ds is the projection of X upon the tangent to the path, etc., and adding the projections of each component,

$$fds=Xdx+Ydy+Zdz.$$

The condition that a field is conservative is that the above differential is *a perfect differential* (so that its integral may be inde-

pendent of the path). In other words, *the components of the field must be the partial derivatives, as to x, y, and z, of a given function,* say $W=F(x,y,z)$, so that we have $fds=dW$. Thus, if the particle starts at (a,b,c), and take any path which ends in (x,y,z), its increase of vis viva will be $F(x,y,z)-F(a,b,c)$.

POTENTIAL.

Conversely, consider any given function, $\phi(x,y,z)$, and let it define a field whose components at (x,y,z) are *the negatives* of its corresponding partial derivatives; and let $\phi(x,y,z)$ be called the potential of the field at (x,y,z).

The field is conservative, because ϕ is simply $-F$ above; also,

$$vdv+d\phi(x,y,z)=0 \text{ or, } \tfrac{1}{2}v^2+\phi(x,y,z)=\text{constant}.$$

In wods, *the sum of the vis viva and potential of a particle moving in a conservative field, is a constant.*

A potential surface is one, every point of which is at the same potential. If a be its potential, its equation is $\phi(x,y,z)=a$. Since a particle will move on a potential surface with constant speed, its tangential acceleration must be zero at every point, i. e., *potential surface is everywhere perpendicular to the lines of acceleration of the field.*

It appears that a particle in motion has on a potential surface, a corresponding speed which it must take if it arrives at that surface. As the potential increases, the corresponding speed decreases, until a surface of zero speed is reached; and we may say that the particle could have acquired its actual speed by having fallen from rest upon the surface of zero speed, to its given position.

EXAMPLES V.

1. Show that all potentials may be changed by a constant without altering the field. Show that the resultant of two conservative fields is a conservative field, and that the resultant potential at any point is the sum of the component potentials at that point.

2. If a point move from $P=(x,y,z)$ to $P'=(x',y',z)$ in a *fixed* direction, lim $NPP'=dq$, is called its *differential variation in that direction,* and lim N $[\phi(x',y',z')-\phi(x,y,z)]=d\phi$ *is the differential variation of the function ϕ in that direction.* Show that if the function be the potential of a field, $-d\phi/dq$ is the component of its acceleration in the direction q of differentiation.

3. With the axis of y upwards, show that the potential of the constant field at (x,y,z) is gy.

4. Show that the potential of the earth's field, at distance r from the center is ga^2/r. This makes zero potential where? If a particle fall with surface of zero speed at infinity, with what speed

would it reach the earth's surface? Where is its surface of zero speed if it reach the earth with greater speed?

5. Show that any central field, varying as a function of the distance only, is a conservative field; also the resultant of any number of such fields.

6. If the free path of a particle which is constrained to lie on a potential surface be called *a geodetic line of the surface,* prove that the radius of curvature of a geodetic line is normal to the surface at every point.

Note. The static definition of a geodetic line is that it is the form assumed by a strained thread constrained to lie on the (perfectly smooth) surface; and the static proof of its fundamental property is analagous to the above dynamical one, viz, the tension of the thread at any point produces a resultant pressure in the direction of its radius of curvature, which, since it is balanced by the pressure of the surface, must be normal to the surface. Geometrically, the geodesic line between two points on a surface is a *minimum line* between the two points, and lying on the surface.

7. If a particle describe a path in any field, freely or by constraint, and its velocity be exactly reversed, it will return over the path with velocity exactly reversed at every point.

Note. Make the free path for the moment constrained, and show that it exerts no pressure in the reverse motion. Observe that in the free path, v^2/R must be the normal acceleration of the field.

8. A free path in a field can be described freely with only one speed at a given point; when a particle describes it as a constrained path, the vis viva at any point differs from the corresponding free vis viva by a constant c, and $2c/R$ is the normal acceleration produced by the pressure of the path.

9. A particle is constrained to move on an ellipse, with one focus an attractive center, the other a repulsive center, the two equal in absolute acceleration. Show that the normal acceleration of the pressure of the path varies as the curvature.

10. Hence show that if a particle be placed at equal distances from the above centers, it will vibrate in a semi-ellipse with the centers as focii.

11. If a line be an axis of symmetry of a field, particles started from a point of the axis with symmetrical velocities will describe symmetrical paths, with symmetrical motions. Every free path which meets the axis at right angles is symmetrical as to that axis.

12. A point of inflection in a free path can occur only at a neutral point (position of zero acceleration) if the field is continuous; it can have no cusp, or other point indicating sudden

change in the direction of motion, except by collision or sudden impulse.

13. An orbit in a central field is concave toward an attractive center, convex towards a repulsive one.

Definition. An *apse* of an orbit in a central field is a point of the orbit which is at a maximum or a minimum distance from the center. The line joining an apse to the center is an *apsidal line;* the distance from the center to an apse is an *apsidal distance.*

14. An apsidal line is perpendicular to the orbit at its apse; if it is an axis of symmetry of the field, it is an axis of symmetry of the orbit (*eg.*, when the acceleration varies as a function of the distance only). An apse cannot be a point of inflection.

15. If a central field varies as a function of the distance only, then the arcs of an orbit between successive apises are equal and similar, and subtend equal angles at the center; alternate apsidal distances are equal so that there are not more than two apsidal distances. The orbit is closed only if the angle between successive apsidals is commensurable with four right angles.

16. Find the equation for determining apsidal distances.

Ans. If $\phi(r)$ be the potential at distance r, $(H/r)^2+2\phi(r)=c$ where c is determined by initial conditions.

17. The periodic time of a planet which is double the earth's distance from the sun is about 1033 days.

18. Prove from the hodograph of a planet that its velocity can be resolved into two constant velocities, c and ec, perpendicular respectively to the radius vector and the major axis.

19. Find the locus of the empty focus and the center of the orbit of a planet which is projected from a given point with given speed.

20. Find the locus of the apses of the orbit above, in any central field.

21. Show that there are two directions of projection of a planet from a given point, so that it will pass through another given point.

II. DYNAMICS

Mass is *quantity of matter.* For homogeneous matter, mass is proportional to volume; a certain volume is *unit mass,* and *the mass of a unit volume is density* so that *mass=density times volume.*

A particle is a quantity of matter concentrated in a geometrical point; it is a fiction which is convenient and almost necessary in the development of dynamics. Its actual function in the dynamics of finite bodies will be considered later.

.. *The Momentum of a particle is the product of its mass into its velocity.* It is consequently a vector quantity, represented by the extension of the velocity vector as many times as there are units in the mass of the particle. Remembering that the momentum of a particle is tangential to the path, it is fully determined by the numerical measure, mv, where m is the mass, and v the speed, of the particle; but momentums add by *the parallelogram law of vectors,* and not by the addition of their numerical values. Thus, the sum of two momentums of 3 and 4 units, varies from 7 to 5 to 1, according as the angle between their directions varies from 0 to 90 to 180 degrees.

The fundamental basis of mathematical dynamics is the postulate that *a particle cannot change its own state of momentum.* Such change must come from *outside causes,* so that if a particle of given mass were wholly unacted upon from without, it would move forever *uniformly in a straight line* (including *permanent rest* as the limiting case of *zero* speed).

Any cause of change of momentum in a particle is named *impulse or force,* according to the character of the change.

An Impulse produces *instantaneous change of momentum.* (The mathematical analogue of a blow). It has magnitude and direction, which are represented by *the change of momentum, i. e.,* the magnitude of the suddenly added momentum, and its direction, are the determining elements of the impulse, so far as it can be mathematically expressed.

A Force produces *gradual and continuous change of momentum, i. e.,* an indefinitely small change in magnitude and direction, in an indefinitely small time.

The Average Force during any time, is represented by the quotient of *the whole change of momentum by the time of change,* i. e., the average change per unit time. It has direction and magnitude as represented by this quotient.

The Force acting at an instant in the middle of a continuous change of momentum, is the limit of the average force (as estimated for an indefinitely small time whose limit is the instant considered). It has magnitude and direction represented by such limit of the average force, and is expressed by components.

Impulses and Forces are plainly vector quantities, being either differences of vector momentums, or the limits of such differences. The entire system of dynamical quantities, velocity, momentum, acceleration, impulse and force, rest fundamentally upon displacements in space, in which a displacement from a point A to a point B *plus* another from B to C, equals a displacement from A to C.

We can take the lines which represent the velocities of the particle in fig. 1, to represent the momentums instead, since there

is no change of direction, but simply a change of measures of length. Thus, OH will be a line of mv units, and OH', a line of $m'v'$ units, so that mv replaces v and $m'v'$ replaces v' in the work of finding the components of acceleration. Hence:

The Tangential Force at time t is $d(mv)/dt = D(mv)$.

The Normal Force at time t is $mvd\phi/dt = mv^2/R$.

If m is a constant, then $D(mv) = mDv = mf$. Also, the normal force is in any case, the mass times the normal acceleration. Hence,

When the mass concerned does not vary with the time, the force acting equals the product of the mass into the acceleration.

A further postulate of mathematical dynamics is that *action and reaction between particles are equal and opposite.*

The preceding postulates and definitions were in point of fact successively deduced from observation and experiment as true in nature, beginning with Galileo, and ending with Newton, who first formulated them as a basis for a mathematical science of dynamics under what he called the *three laws of motion.* The most crucial tests that these laws are in accord with nature are the many verifications of the Newtonian Dynamics in Astronomy (see Encyclopedia Britannica, article "Mechanics," sections 3 and 4). The theory has been applied successfully in the development of all departments of Physics and even of Chemistry, and the theory of all natural phenomena is becoming more and more a purely dynamical science.

The true measure of mass is dynamical. That is, mass is a property of *inertia* in matter, in virtue of which external action is required to change its state of rest or of uniform motion in a straight line. Thus, equal masses are those which under the same external actions, observe the same changes in velocity. This is reduced, in practice, to the case of external actions in equilibrium. Thus, equal masses will in vacuo extend a spring balance equal amounts; or, placed in the opposite pans of an arm balance, will not disturb its equilibrium; or, coming together with equal and opposite velocities, and adhering, will be brought to rest; etc.

FINITE BODIES.

A finite body can be divided by imaginary surfaces into indefinitely small elements. The surfaces are determined by the coordinate system, and are such that along any surface, one of the coordinates of the system is a constant. Thus, in the Cartesian systems, x, y, z, the surfaces are planes parallel to the coordinate planes. In the *polar* system, r, θ, ϕ, the surfaces are spheres about the pole as center, circular cones about the initial line as axis, and

planes through the initial line. In the cylindrical system, r, θ, z, the surfaces are cylinders about the axis of z, planes through and planes perpendicular to, that axis.

Consider the element which contains a given point P, and ultimately reduces to that point. So long as this element of the body has not vanished, it has form, volume, and mass; and we may construct a similar element in form and distribution of mass, in so large a proportion to the vanishing element that it remains finite, and approaches a limit, called *the differential element at P*. The limits of the sums of all the elements of volume and mass of the body are plainly the whole volume and mass. But these are the integrals of the corresponding differentials, by definition of integration.

The differential element pertaining to a point P, is rigorously a particle of mass dm, concentrated at P. In fact, its distance from any fixed point, line, or plane, corresponds, in the summation of the indefinitely small elements of the body, to the distance of some point of the small element to which it corresponds, and all of these points become in the limit, the single point P. Also, in considering the forces on a body, they reduce to the systems in which the forces on each element are those which produce its actual motion as a separate body, and in proportionalizing the forces on an element with the element, they become ultimately forces acting at P, whose resultant is the differential force on the particle dm. Thus, the rotational effects of the forces on the small elements disappear in the limit.

The integral of particles over a line is a *filament;* the integral over a surface, is a *lamina*. Forces on the several particles must, of course, be integrated in respect to their components in *fixed directions,* and in respect to their moments about *fixed axes* in order to obtain their resultant effects upon the body as a whole.

The force on a particle is *its mass times its acceleration*. The mutual actions of a system of particles are called *internal forces;* but when the integration of a component or moment of force on a particle extends to every particle of the system, the effects of the internal forces mutually cancel, since action and reaction are equal and opposite, so that the integration obtains simply the corresponding component of *resultant external force,* or *resultant moment of external force,* as the case may be.

KINETIC ENERGY AND WORK

If m be the mass, and v the speed of a particle, its kinetic energy is $\frac{1}{2}mv^2$, or *the product of mass into vis viva*. The kinetic energy of a body is $\int \frac{1}{2}v^2dm$, extending to every point of the body.

The work done by a *constant* field of force in any displacement of the body is, *force times displacement times cosine of angle between them,* it may also be expressed as, *force times component of displacement in its direction,* or *displacement times component of force in its direction.*

The sign of the work is *positive* or *negative* according as the angle between force and displacement is *acute* or *obtuse;* it is *zero* if the angle is a *right* angle.

The work done in two successive displacements is the work done in the resultant displacement; and the work done by the resulant of two constant fields is the sum of the works done by each.

The work done by a field which varies from point to point must be considered with reference to the path. We divide the path into elements so small that the field is nearly constant for the displacement of any element; and so *rigorously constant* in the differential field corresponding to the point to which the element ultimately reduces. In other words, the exact work done is *the integral over the path of the differential work at each point*—remembering that this stands for nothing else but the limit of the sum of the works done along each element under the more and more nearly rigorous hypothesis of the constancy of the field over the element.

If W denote the work of the field on a particle which describes the path AP (fig. 1) in the field, we have, $dW = mfds = mvdv$. Thus, if V be the speed at A, this gives, by integration, $W = \frac{1}{2}mv^2 - \frac{1}{2}mV^2$.

In words: *The work done by a field upon a particle moving over any path equals its increase of kinetic energy.*

This will include also the work done by a medium, friction, and any other forces, whose tangential components are included in mf above.

This gives a new definition of the potential at P in a conservative field, viz., the work the field will do in bringing a particle of unit mass from P to the surface of zero potential; or, the work that must be done against the field in bringing it from zero to its given potential.

TRANSLATION OF A RIGID BODY

In what is called translation of a body, all points describe equal distances s, in parallel straight lines. The speed $v = Ds$ of any point, is *the speed of the body,* and Dv is its *acceleration.*

The momentum is $\int vdm = mv$.

The kinetic energy is $\int \frac{1}{2}dm.v^2 = \frac{1}{2}mv^2$.

The resultant force is $\int dm.Dv = mDv$. Since the forces on all

the particles are parallel and proportional to their masses, this resultant force passes through the *center of gravity* of the body.

ROTATION OF A RIGID BODY

In this motion, all points describe circular arcs of equal radian measure θ about a common axis, so that if r be the perpendicular distance from the axis to a given particle P, the arc distance described by the particle is $s=r\theta$; r varies with the position of the particle, but not with the time; θ varies with the time, and is the same for all particles. $D\theta,=w$, say, is *the angular velocity* of the body, and Dw is its *angular acceleration.*

The velocity of the particle P at distance r from the axis is $Ds=rD\theta=rw$, perpendicular to r.

The momentum of the particle is therefore $w.rdm$, perpendicular to r, so that its moment of momentum about the axis is $w.r^2dm$. Let I stand for $\int r^2dm$, extending to every particle of the body. Then,

Moment of Momentum about the axis is Iw.

Similarly, from $\int \frac{1}{2}v^2dm$, and $v=rw$, *the Kinetic Energy is* $\frac{1}{2}Iw^2$.

The tangential force on P is, $dm.Dv=rdm.Dw$, and the normal force is, $dm.v^2/r=rdm.w^2$. Since the latter intersects the axis, its moment about the axis is zero. Thus, the moment of the force on the particle P, about the axis is, $r^2dm.Dw$. Hence,

The resultant Moment of Force about the axis is, IDw.

One should observe the parallel between translation and rotation. In the former, v and Dv, are *linear* velocity and acceleration, and mv, $\frac{1}{2}mv^2$, mDv, are the momentum, kinetic energy, and resultant force. In the latter, w and Dw, are the angular velocity and acceleration, and Iw, $\frac{1}{2}Iw^2$, IDw, are the moment of momentum, kinetic energy, and moment of force. The inertia factor m in translation, is replaced by the inertia factor I in rotation, which is consequently called *the moment of inertia of the body about the axis.* From its definition, the moment of inertia depends solely upon the distribution of mass about the axis, and, is made up of *the sum of the moments of inertia of its component parts.* The distance from the axis at which a body can be concentrated with the same moment of inertia is called its *radius of gyration about the axis.* Thus $I=mk^2$, where k is the radius of gyration.

It is shown in the calculus that if the axis, *originally at the center of gravity,* be moved parallel to itself a distance a, the square of the new radius of gyration is k^2+a^2.

THE NORMAL FORCE OF REVOLUTION.

Since the normal forces meet the axis, they are balanced by, and determine the stress on, the axis. It is necessary to consider coordinate axes, and moments about them, for which refer anew to the topics on moments (page 17). Take the axis of revolution as axis of z; let the plane of xy contain the center of gravity of the body; and let P be any point of the body whose coordinates are (x, y, z), mass dm, and distance $PZ=r$ from the axis of z. (With some constructions by the student, named portions of fig. 2 will answer as a diagram).

Let PN (on PZ) be the normal force on P. We have shown above that $PN=rw^2dm$, or in x, y, z, components,

$-w^2xdm, -w^2ydm, 0$.

Introduce the balancing forces $OM, -OM$, at O, where $OM=PN$. Thus, the force PN on the body is replaced by the equal and parallel force OM, acting at O, and the couple, $PN, -OM$, which is fuly expressed by its moment $2OPN$. Finding the components of this moment from the components of PN and the coordinates of P, $(Yz-Zy$, etc.) they are, $w^2dm.yz, -w^2dm.zx, 0$.

Do the same for every particle of the body, and integrate the several components of force and moment, extending to every particle of the body. The coordinates of the center of gravity are say, $(x', y', 0)$, so that ($m=\int dm$ being the whole mass),

$\int xdm=mx', \int ydm=my'$

Also let $\int yzdm=A, \int zxdm=B$. These integrals are called *products of inertia of the body about the axis;* and when they are both zero, the axis is called *a principle axis of the body for the point O.* Using these integrals, we thus resolve the normal forces of revolution into a single force acting at O, and a couple, whose components are:

The force, $-w^2mx', -w^2my', 0$.

The couple, $w^2A, -w^2B, 0$.

Plainly, the resultant normal force is the same *as if the whole mass of the body were concentrated at its center of gravity,* so that it is zero only when the axis of revolution passes through that center of gravity. Also, the couple is zero only when the axis is a principle axis for the point O. In words, *there is no stress on an axis of revolution of a body (direct or twisting) when and only when the axis is a principal axis of the body through its center of gravity.*

THE COMPOUND PENDULUM.

Any body revolving about a fixed horizontal axis is *a compound pendulum.* Let $OG=a$, be the distance of the center of gravity from the axis, θ the angle between OG and the downward

vertical, OL, and I, the moment of inertia about the axis. Computing moment of the force (gravity) about the axis,

$$I.D^2\theta = -mga \sin\theta.$$

Compare this with the equation for the simple pendulum of length l, at the same angle, at any time t, $l.D^2\theta = -g\sin\theta$, and we find that $l = I/ma$, which is called the length of *the equivalent simple pendulum.*

Produce OG to C, making $OC = I/ma$; C is called *the center of oscillation as to the center of suspension O.* Let k be the radius of gyration about a parallel axis through the center of gravity G; then $I = m(a^2 + k^2)$. Thus $OC = a + b$, where

$$b = GC = k^2/a.$$

Conversely, if C be made center of suspension with a parallel axis, its center of oscillation will be on CG produced, at a point which is a distance k^2/b, or a beyond G, which makes it the point O. Thus, *centers of suspension and oscillation are interchangeable.*

EXAMPLES VI

1. Prove that the whole work done in raising a system of bodies through different heights is the same as raising the entire weight through a height equal to that which their center of gravity is raised.

2. Find the work performed in moving a ton 100 yards on a uniformly rough horizontal road, the coefficient of friction being one-tenth.

3. Show that the same work is expended in drawing a body up an inclined plane, subject to friction, as would be expended by drawing it along the base and raising it vertically upward.

4. If two particles attract each other directly as the product of their masses, and inversely as the square of the distance between them, find the work done, when they have moved from an infinite distance apart to the distance r.

5. Find the potential outside, upon, and within, a spherical shell, of uniform mass per unit area. Derive thence its attraction upon a particle in the same situations.

6. Show hence, that a solid spherical mass, whose density at any point varies only with the distance from the center, attracts an outside mass as if its whole mass were concentrated at the center.

7. Determine the law of force wthin a tube along a diameter of the earth.

8. A flat pivot presses against a rough plane; find the work done against friction in one revolution of the pivot.

9. A ball moving with a velocity of 1,000 feet per second has

its velocity reduced by 100 in passing through 1.9 inches of wood; what is its maximum penetration in the same material?

10. A rectangular log lies on the floor of a car, in direction perpendicular to the sides. If it be prevented from slipping, determine the speed of the car, such that if suddenly stopped, the log shall be just overturned.

11. A sling is formed by fixing the ends of an elastic cord (natural length $2a$) to the prongs of a stick. A bullet of mass m is placed at the middle of the cord and drawn back until the cord is of length $2b$. Determine the velocity which the sling will give to the bullet.

12. An elastic cord, absolute tension k, is attached to a mass m, resting on the ground. The free end of the cord is raised vertically until the mass is just lifted; find the work done.

13. An indefinitely long chain, mass m per unit length, is coiled on the ground. A ball of mass km attached to the end of the chain is projected vertically with speed V; determine its speed at any time and the height to which it rises. (Note. The mass is here variable). Consider the same projected on a smooth horizontal plane.

14. A rod of uniform density is let fall from a horizontal position, one end being attached to a fixed horizontal axis. Find its angular velocity when vertical.

15. A weight is attached to a cord which is wound round a homogeneous cylinder with axis horizontal. The weight falls, turning the cylinder. Find its speed at any time, if the cord is weightless, and also allowing for its weight.

16. Find the speed acquired by the center of a hoop in rolling down an inclined plane of height h.

17. A homogeneous cylinder or sphere, rolls down an inclined plane without slipping. Investigate its motion.

18. Find the center of oscillation of a uniform circular plate, with respect to any point as center of suspension. Also, the position of the center of suspension for minimum time of small oscillation.

19. Find the center of oscillation of a homogeneous sphere, about a tangent, and compare its time of oscillation with that of a spherical shell of equal diameter, about a tangent.

20. A pendulum is an elliptic disc, axes four and two feet, point of suspension a focus; determine its period and center of oscillation.

21. A bent lever ACB rests in equilibrium when AC is inclined at an angle e to the horizontal; show that if this arm be

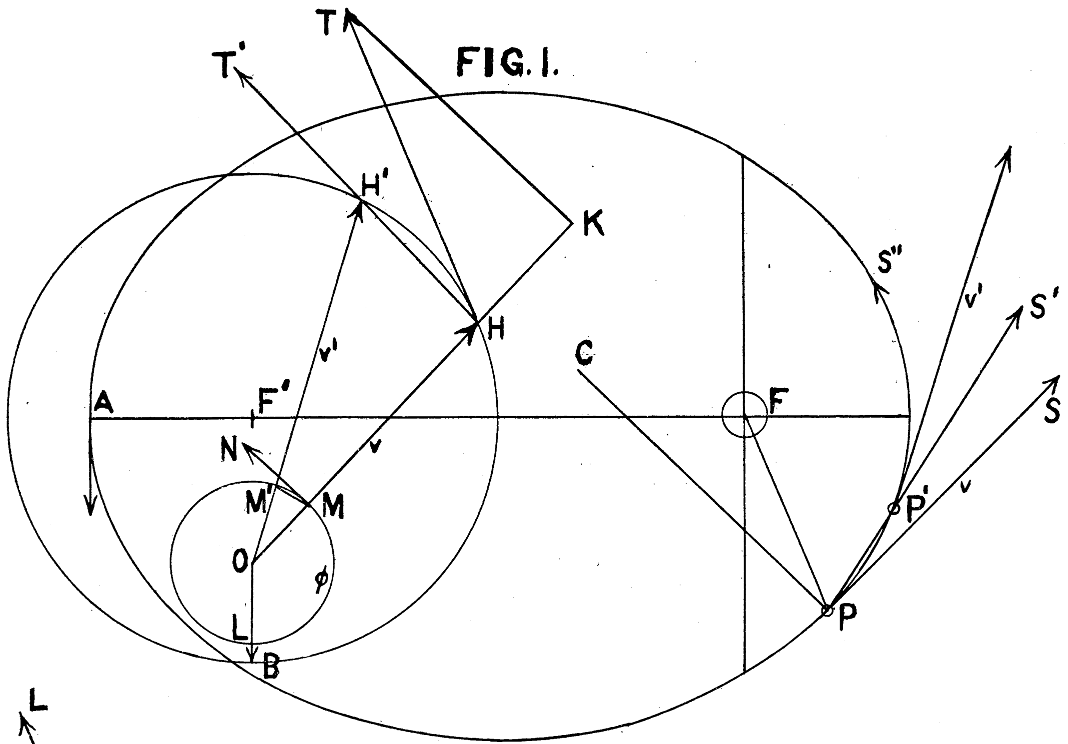

FIG. 1.

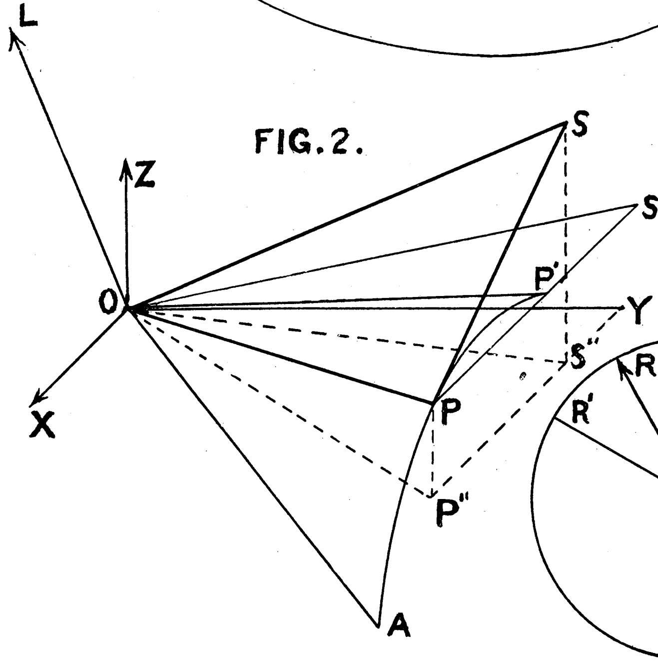

FIG. 2.

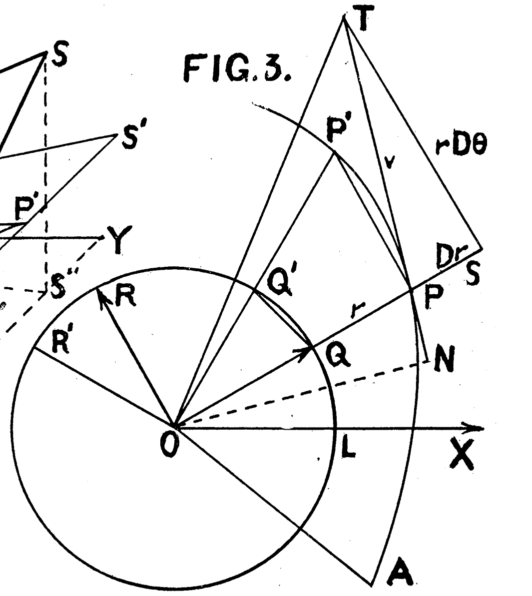

FIG. 3.